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Nonadiabatic corrections to the adiabatic Efimov potential

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Abstract. Our discussion of the Efimov effect in an adiabatic representation is completed here by examining the contribution of all the nonadiabatic corrections. In a previous article by Fonseca et al, the lowest order adiabatic potential was derived in a model three-body problem, which showed the critical $-1/x^2$ behavior for large x, where x is the relative distance of two heavy particles. Such a potential can support an infinite number of bound states, the Efimov effect. Subsequently, however, we showed that the leading nonadiabatic correction term $< K_x >$, where K_x is the heavy particle relative kinetic energy operator, exhibited an unusually strong $1/x$ repulsion, thus nullifying the adiabatic attraction at large values of x. This pseudo-Coulomb disease (PCD) was speculated to be the consequence of a particular choice of the Jacobi coordinates, freezing both heavy particles. It is shown here that at large x, the remaining higher-order correction $< K_x G^{\Sigma} K_x >$ cancels the PCD of $< K_x >$, thus restoring the adiabatic potential and the Efimov effect. Furthermore, the nonadiabatic correction is shown to be at most of order $1/x^3$. This completes the discussion of the Efimov effect in the adiabatic representation. Alternatively, a simple analysis based on the static picture is presented, for comparison with the adiabatic procedure. The non-static correction is of order $-1/x^2$; this suggests that the adiabatic picture may be preferred in obtaining the Efimov potential.

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1 Introduction

One of the exotic behaviors of many-particle systems is manifested in the property of the Efimov effect [1], which states that, in an interacting three-particle system, when a subsystem of two particles supports a zero energy 'bound state', the total system of three particles can have an infinite number of bound states, independent of details of interparticle interactions so long as they are attractive and relatively short-ranged [2]. This property is usually shown by deriving an effective three-body potential to be of the 'dipole-type', i.e. inversely proportional to the square of the distance separating the heavy particle pair. Then, the number of bound states is N $\simeq tr(G_0 U) = \int_{r_0}^{A} r^2 dr (1/r)(1/r^2) \simeq \ln A \to \infty$ as $A \to \infty$, where A is the scattering length of the pair that is weakly bound (near zero energy).

A three-particle model was constructed by Fonseca et al. [3] to study the Efimov effect. For convenience, an adiabatic representation was adopted in deriving directly an effective potential that has the required property. The model given in [3] describes two heavy particles (1 and 2) coupled to a lighter particle (3), and the interactions between the light and heavy particles are assumed to be of a separable form. This model in the lowest adiabatic approximation is then essentially a two-body problem and is exactly solvable. When the light particle binding energy approaches the value zero, the resulting potential energy for the heavy particles showed the critical asymptotic behavior $U_a^{ad} \sim -1/x^2$ at large x. Here, $\vec{x}$ denotes the relative coordinates of two heavy particles 1 and 2, which are fixed during the calculation of the adiabatic energies and wave function.

However, the original proof by Fonseca et al. [3] was incomplete because the nonadiabatic correction (NAC) terms were not examined, as to their large x behavior. In ordinary cases where the two-body binding energy is negative, the NAC is in general of a much shorter range, and thus causes no difficulty in so far as the Efimov effect is concerned. But, in case of a zero-energy bound state, the question of NAC can be critical.

In a recent report [4], we examined the first nonadiabatic correction $< K_x >$ to the adiabatic potential. Surprisingly, it was found that the leading correction behaved as $+1/x$ for large x, thus essentially nullifying the Efimov property. This is obviously a serious blow to the adiabatic potential picture, because the long range behavior of the repulsive 1/x potential, while moderated by the adiabatic

mass ratio m/M, will eventually over-compensate the attractive potential $U_a^{ad}(x) \propto -1/x^2$ term. We made several conjectures as to possible causes of this pseudo-Coulomb disease (PCD) in [4].

Inspite of this spurious PCD, the adiabatic picture is intuitively appealing, especially when the binding energies involved are much smaller than the individual masses of the particles involved. At large distance, the relative kinetic energies must also be very small. The concept of adiabaticity can also be applied to many interesting situations that involve particles at very low temperatures. Therefore, it is essential that the problem of the PCD must be resolved by systematically considering all the nonadiabatic corrections, such that preferrably the theory may be applied with confidence at the lowest adiabatic approximation.

The present report is a continuation of the earlier study of Efimov effect, discussed first within the adiabatic representation. We show that the PCD difficulty disappears when all the higher order NAC are properly taken into account, thus restoring the original lowest order result. This therefore completes the proof of the Efimov effect in the adiabatic representation. The result is consistent with the general view that the Efimov effect should, if valid, be proven, independent of particular representation chosen or approximations introduced, so long as a complete set of basis states is properly taken into account. Difficulties of the PCD feature seem to appear because the problem is only partially treated.

In order to further clarify the physics involved in the adiabatic approach, we also consider the Efimov effect in the static picture, which is the opposite of the adiabatic approach. The higher order corrections in this case are examined; it turned out that no stringent limitation on mass ratios is needed. This is shown in Appendix A and Appendix B.

2 Preliminary discussion

We first review previous results, mainly to state the problem and fix notations. We limit our discussions to the original model of Fonseca et al. [3] for a three-body system with separable potentials. The Jacobi coordinates $(\vec{x}, \vec{y})$ they adopted in [3] were the conventional 'molecular' ones in which the relative coordinates $\vec{x}$ between two heavy particles 1 and 2 are fixed in space during the calculation of motion of light particle 3, whose coordinates relative to the center of mass of 1 + 2 are denoted by $\vec{y}$. The total Hamiltonian is given by

$$H = K_x + H_{ad}(\vec{y};\vec{x}), \qquad H_{ad} = K_y + V_{13} + V_{23} \quad (2.1)$$

where the potential V_{12} is omitted for simplicity. Note that K_y is defined in terms of $\mu \equiv \mu_3 = 2mM/(m+2M) = m/(1+\varepsilon/2)$, where $\varepsilon = m/M$. Depending on the reduced mass taken for K_y, the new operator $K_y' = K_y(\mu_3/\mu_2)$ may be used in H_{ad}'. Then, a term $(1-\mu_3/\mu_2)K_y$ may be added to K_x of (2.1) for a finer adjustment, where $\mu_2 = m/(1+\varepsilon)$. In [2], the potentials were chosen to be of a nonlocal form

$$V_{13} = -\lambda f(13) f(13'), \text{ with } f(r) = \exp(-\gamma r)/r \text{ etc.} \quad (2.2)$$

In the following we use $\mu = \mu_3$ and hence with K_y, H_{ad}, and K_x; we remark that an alternate theory with K_y', H_{ad}', and $K_x' = K_x + (1-\mu_3/\mu_2)K_y$ is possible.

Solutions of the adiabatic subsystem described by H_{ad} for the motion of light particle are defined by

$$[H_{ad} - \eta_\alpha(x)]\phi_\alpha(\vec{y};\vec{0}), \text{ with } (\phi_\alpha, \phi_\beta)_{\vec{y}} = \delta_{\alpha\beta} \quad (2.3)$$

The adiabatic energy η_α then gives the adiabatic potential $U_\alpha^{ad}(x) \equiv \eta_\alpha(x) - E_\alpha \to 0$ for $x \to \infty$ and E_α, is by definition the asymptotic value of the adiabatic energy. The normalization condition in (2.3) is valid for $E_\alpha < 0$, but needs care when this energy approaches zero. (See the discussion in subsect. 2d.).

In order to examine systematically all the nonadiabatic corrections in this paper, we first formulate the full problem in the adiabatic state representation. The solution of the full problem $\Psi(\vec{y},\vec{x})$ may be expanded as

$$\Psi(\vec{y},\vec{x}) = \sum_\alpha \phi_\alpha(\vec{y};\vec{x})\chi_\alpha(\vec{x}) = \Pi\Psi + \Sigma\Psi \quad (2.4)$$

where $\Pi = |\phi_\alpha><\phi_\alpha|$, with $\alpha = a$ and $\Sigma = 1 - \Pi = \Sigma^+ = \Sigma^2$ and $\Pi\Sigma = 0$. These are projection operators in the $\vec{y}$ variable, with $\vec{x}$ fixed. They span the adiabatic function space of variable $\vec{y}$, with $\vec{x}$ again a fixed parameter. The above definition for the projection operator Π is valid when $E_\alpha < 0$. The case of interest here involves $E_\alpha \to 0$, which requires special caution. That is, at zero energy, the distinction between Π and Σ is less clearcut. (See the discussion in subsect. 2d and in Appendix B.)

The substitution of (2.4) into the equation of motion with the total Hamiltonian H gives a set of coupled equations

$$\Pi[H-E]\Pi\Psi = -\Pi K_x \Sigma\Psi \quad (2.5a)$$

$$\Sigma[H-E]\Sigma\Psi = -\Sigma K_x \Pi\Psi \quad (2.5b)$$

The formal uncoupling of the equations gives exact equations for $\Pi\Psi$ and $\Sigma\Psi$. We are interested here in the Π part, which is given as

$$\Pi[H - E + K_x G^\Sigma K_x]\Pi\Psi = 0 \quad (2.6)$$

where

$$G^\Sigma = [\Sigma(E-H)\Sigma]^{-1}. \quad (2.7)$$

One must be careful in defining this Green's function, because $\Sigma G \Sigma \neq G^\Sigma = \Sigma G^\Sigma \Sigma$. That is, an inversion of the operator (E−H) does not commute with the projection Σ. Explicitly, using the adiabatic basis set defined by (2.3), we have for (2.6),

$$[K_x + <K_x> + <K_x G^\Sigma K_x> + U_a^{ad} - E_a']\chi_a(\vec{x}) = 0. \quad (2.8)$$

where we used $| \ > = |\phi_a >$. This equation is exact, with no approximations yet introduced. A similar reduction can be carried out for $\Sigma\Psi$ as well, with the resultant set of coupled equations for χ_α, $\alpha \neq a$. The term $\int \phi_a^+(\nabla_x\phi_a)d\vec{y} \bullet \nabla_x = 0$ by the normalization of ϕ_a, and $E'_a \equiv E - E_a$. The leading nonadiabatic correction term $< K_x >$ is defined by

$$U_a^{na-I} \equiv < K_x > = \int \phi_a^+[\vec{K}_x\phi_a]d\vec{y} > 0 \qquad (2.9)$$

where the square bracket denotes the operation of the differential operator K_x limited to functions to its right, but within the bracket. By definition of the kinetic energy operator, this term is positive for all x.

2.1 Adiabatic approximation

Fonseca et al. [3] constructed a separable potential model which can be solved exactly in the adiabatic approximation (2.3); that is, all the averaged K_x-dependent terms in (2.8) are omitted. In particular, they showed that

$$U_a^{ad}(x) \to -1/x^2, \text{ as } x < A \to \infty, \qquad (2.10)$$

where A is the scattering length of the pair $(1+3)$ or $(2+3)$. This then immediately establishes the fact that in principle, there are an infinite number of bound states for the three-particle system when the two-particle subsystem supports a zero-energy bound state. However, obviously, the nonadiabatic correction terms $< K_x >$ and $< K_x G^\Sigma K_x >$ have to be examined for their large x behavior, where $< >$ denotes an average over state ϕ_a. We note that $< \vec{K}_x G^\Sigma \vec{K}_x > = < \vec{K}_x G^\Sigma \vec{K}_x >$, where explicit directions to which K_x is to operate are indicated.

2.2 First nonadiabatic correction potential

The first order term in K_x in (2.8) was studied recently in [4], and it was shown that

$$< K_x > \to + constant/x \text{ as } x \to \infty, \qquad (2.11)$$

where the constant coefficient is of the order of the adiabatic mass ratio, which is a small number. Nevertheless, this result was disastrous, since (2.11) can easily overshadow the large x behavior of (2.10), and thus nullify the earlier proof of Fonseca et al. By contrast, in the conventional theory with one or more negative energy bound pair states, $E_b < 0$, the potential U_a^{ad} approaches zero faster than $1/x^3$ at large x. Except for a small region in the reduced mass space, the potential $U_a^{ad} + < K_x >$ failed to exhibit the Efimov effect. This is the pseudo-Coulomb disease (PCD). For later discussion, it is important to emphasize that, as stated in (2.9), the asymptotic form of this correction is repulsive.

2.3 x-power counting

To further understand this result, we analyze (2.3) and the adiabatic function $\phi_a(\vec{y};\vec{x})$. In terms of states generated by $h_2 = K_y + V_{23} = H_{ad} - V_{13}$. as $h_2\psi_n^{(2)} = \varepsilon_n\psi_n^{(2)}$, we can write

$$\phi_a(\vec{y};\vec{x}) = [\psi_a^{(2)} + \xi_a]/N(x) = \phi_a^P + \phi_a^0, \qquad (2.12)$$

where $\mathrm{P} = |\ \psi_a^{(2)}\ )(\ \psi_a^{(2)}\ |$ and $\mathrm{Q} = 1 - \mathrm{P}$. Here, the operator K_y is still defined in terms of the reduced mass μ_3. Again we caution the meaning of P and the normalization of $h_2 = K_y + V_{23} = H_{ad} - V_{13}$. (For the details of this analysis, see Appendix A, where a slightly different set of coordinates are used.) As this function is normalized for each fixed value of $\vec{x}$, we set

$$(\phi_a|\phi_a)_y = 1, \quad (\psi_a^{(2)}|\psi_a^{(2)}) = 1, \quad (\psi_a^{(2)}|\xi_a)_y = 0. \qquad (2.13)$$

Thus, we have

$$N^2 = (\xi_a|\xi_a) + 1. \qquad (2.14)$$

First we show that $(\xi_a|\xi_a) \to 1/x$. From perturbation theory and short-range interaction for the V's, we have

$$\xi_a \simeq G_0^Q V_1 \psi_a^{(2)} \simeq \gamma \int d\vec{y}\delta(1/y)^2 \propto 1/x^2, \qquad (2.15)$$

where we indicated only the x-dependence, and $V_1 = V_{12} + V_{13}$. This behavior was explicitly shown in [4]. Then, at large x,

$$(\xi_a|\xi_a) \simeq (1/x^2)^2 \bullet x^3 \simeq 1/x \qquad (2.16a)$$

$$N^2 \to 1 + (1/x) \to 1 \text{ and } \xi_a \to (1/x^2), \qquad (2.16b)$$

where, in (2.16a), the last x^3 factor comes from the volume integral $d\vec{y}$ that stretchs out to large x. We also recall that in (2.16b) the function ξ_a is by definition (2.12), in the Q space and is orthogonal to ψ_a. Now, we can estimate

$$\nabla_x\phi_a = N^{-2}[-\psi_a^{(2)}dN/dx - Nd\xi_a/dx] \to N^{-2}[-\psi_a^{(2)}/x^2 - N\xi_a/x], \qquad (2.17)$$

where $\mathrm{dN/dx} \simeq 1/x^2$. According to (2.15), the last term in the bracket in (2.17) behaves as N/x^3 at large x and is in the Q space, since at large x, $d\xi_a/dx \propto -\xi_a/x$. Therefore, the earlier result for the first nonadiabatic correction U_a^{na-I} behaves as

$$U_a^{na-I}(x) \simeq (\nabla_x\phi_a|\nabla_x\phi_a) \propto x^3/x^4 \simeq +1/x, \qquad (2.18)$$

which is the PCD. The detailed calculation of U_a^{na-I} given in [4] is thus reproduced. We note that this PCD comes from the derivative of the normalization N associated with the first term in ψ_a that is proportional to ϕ_a. The second term in ϕ_a gives rise to $1/x^3$ contribution, and this part is important in the estimation of higher-order correction discussed in the next section.

The schematic derivation given here, with explicit x dependences, will be adopted in the next section for the higher order corrections which are much more complicated. Inspite of its simplicity, the x-power counting procedure developed above is rigorous, as much of the results are also substantiated by explicit, and often tedious, calculations.

2.4 Normalization problem

One critical remark is in order, concerning the projection operator Π and the normalization of ϕ_a. In the ordinary case where the binding energy associated with ϕ_a is negative definite, the point spectrum of ϕ_a is distinct from the rest, and the projections Π and Σ are unambiguous. But, when the state ϕ_a represents a zero-energy bound state, the situation is more delicate, because the state is not normalizable, and it degenerates with the dense continuum spectra which lie just above it. Therefore, in defining $\Pi^2 = \Pi$, we will assume an $-\varepsilon = 0^+$ added to the state, and at the end let it approach zero. One consequence of this is that the presence of Σ for example must be treated with caution; in so far as K_x is concerned, the distinction between Π and Σ is minimal in the limit $E_a \to 0$.

One additional point on the form (2.18) should be noted. The quadratic form involving $([\nabla_x \phi_a] \bullet [\nabla_x \phi_a])$ comes originally from the expression $(\phi_a \nabla^2 \phi_a)$ by partial integration and by neglecting the surface term. For a zero-energy bound state, this term is a non-zero constant which is proportional to the square-root of the binding energy $\varepsilon_a \simeq 1/A^2 \to 0$. In fact, if we evaluate directly,

$$-(\phi_a|\nabla^2\phi_a) \simeq -(\phi_a^P|P[1/x^3 + 1/x^4 + \ldots]) + \ldots \\ \to (1/\varepsilon_a)\Theta(\varepsilon_a) + 1/x + \ldots \qquad (2.19)$$

where $\Theta(z) = 0$ for $z < 0$ and $= 1$ for $z > 0$, and $\varepsilon_a \leq 0$. We simply dropped this first term in the discussion thus far. Similar remarks apply also for (2.3) and (2.13). Therefore, this point should be kept in mind when we manipulate quantities which are presumably well-defined for $\varepsilon_a < 0$, but becomes less stable as ε_a approaches the value zero.

2.5 Sources of the PCD

We conjectured in [4] that the PCD may arise from the particular choice of the coordinates $(\vec{y}, \vec{x})$. This allows the kinetic energy operator for the three-particle system to be separable in $\vec{x}$ and $\vec{y}$ variables, but introduces an unphysical constraint in that particles 1 and 2 are fixed in space when a bound state at zero energy is produced for the light particle 3. More precisely, the pair $(1 + 2)$ moves slowly about the total center of mass as particle 3 is allowed to revolve into a zero energy bound state. This picture is reasonable only when the masses of 1 and 2 are much larger than that of particle 3. More importantly, properties of the solution at very small energies require special attention; when binding energy and kinetic energy involved are both very small, intuitively appealing physical pictures are no longer valid. Thus, either a new coordinate system may be chosen, or the additional correction term $<$ KGK $>$ in (2.8) must be included to possibly cancel the spurious PCD. In the next Sect. 3, we examine this second possibility. The first point raised above will be analyzed in separate reports [5, 6]

3 Higher order nonadiabatic correction terms

The previous work showed the important result that, in the presence of a near zero energy two-body bound state, the first non-adiabatic correction gives not only a spurious long range potential, but makes the range of applicability of the theory seriously narrowed down. This is obviously a very unsatisfactory situation, especially in view of the fact that the adiabatic representation provides a complete set of basis states. Evidently, it is suspected that the rest of the nonadiabatic correction terms may not converge, and yield a contribution to the overall potential which is as strong as the first order correction, in such a way to compensate each other. It is the purpose of this section to give a proof that indeed this is what happens; all the nonadiabatic corrections taken together give an improved asymptotic behavior, and restore the adiabatic picture.

We evaluate the remaining correction term $<$ KGK $>$. First summarize the total effective potential that enters the exact equation (2.8). It is given by

$$U_a = U_a^{ad} + U_a^{na-I} + U_a^{na-II} \equiv U_a^{ad} + U_a^{na}, \qquad (3.1)$$

where U_a^{ad} is the adiabatic potential generated by H_{ad}, and where

$$U_a^{na-I} = < K_x > > \ 0 \qquad (3.2a)$$

$$U_a^{na-II} = < K_x G^{\Sigma} K_x > \\ = < K_x \Sigma [\Sigma (E_a - K_x - H_{ad}) \Sigma]^{-1} \Sigma K_x > < \ 0 \qquad (3.2b)$$

where again $| \ > \equiv \ |\phi_a >$. The first nonadiabatic (na) correction term in (3.2a) was discussed in Sect. 2, and (3.2b) contains the remaining correction which we discuss below. Before studying this correction, we note that, when the full Hamiltonian H does not support any bound states except the one which corresponds to the Efimov effect, then the Σ subspace operator is such that the correction $U_a^{na-II} < 0$. This is an important property, as in the following we are interested in the possibility of cancellation between the leading correction U_a^{na-I} (3.2b) in so far as the asymptotic parts are concerned; the first correction is positive and the second term is negative everywhere in x.

Evaluation of the correction (3.2b) is complicated by the presence of the full Green's function in the Σ space, where $G^{\Sigma} = \Sigma G^{\Sigma} = G^{\Sigma} \Sigma$. It is convenient to write the Green's function in (3.2) into two terms, as

$$G^{\Sigma} = [\Sigma(-K_x)\Sigma]^{-1} + [\Sigma(-K_x)\Sigma]^{-1} \Sigma (H_{ad} - E_a) \Sigma G^{\Sigma} \\ \equiv G_K^{\Sigma} + G_{KK}^{\Sigma}. \qquad (3.3)$$

Therefore, U_a^{na-II} further splits into two terms. We now consider their contributions to the potential, each separately. We show that the contribution from the first order correction (3.2a) is cancelled by a part of (3.2b) that comes from the first term in (3.3), while the second term in (3.3) contributes at most to the order $1/x^3$ in U, as will be discussed fully below.

3.1 First nonadiabatic correction

The first term in (3.3) may be approximated by dropping the projection operator Σ, because (i) in the present special case where only one zero-energy bound state is present, the spectrum of H_{ad} is represented effectively by the $\delta(\vec{y}-\vec{y}')$ in the G_K^Σ. Any correction to this approximation should be of short-range with respect to x. That is, essentially the full spectrum of H_{ad} is present in G_K^Σ, and thus we may set $G_K^\Sigma \to G_K = -1/K_x$. This is a critical step in the proof of cancellation. (ii) The separation (3.3) preserves the 'positivity' of the first term, and probably the positivity of the second term as well. (iii) Furthermore, at large x the operator $\Sigma \to Q$ as $x\ (< A) \to \infty$, where Q is the projection without the distortion. (A slight complication here, due to the fact that the center of mass correction must be incorporated in order to have the above identification valid. We ignore this problem at the moment.) As a result, we have $\Sigma \to Q$ in the asymptotic region. Therefore,

$$\begin{aligned} &< K_x > + < K_x \Sigma G_K^\Sigma \Sigma K_x > \simeq \\ &< K_x > \ + \ < K_x(-K_x)^{-1} K_x > \simeq 0 \\ &\text{to order } 1/x^3, \end{aligned} \tag{3.4}$$

where we used the critical fact that, as far as the asymptotic properties of K_x. is concerned, the distinction between Π and Σ is not crucial in the limit $E_a \to 0$, because the zero-energy bound state (ZEBS) represented by Π is 'half-way' in Σ, and vice versa. (See the discussion in Appendix B.) Moreover, the operator K_x does not carry information on the spectrum of H_{ad}, so that we replaced G_K^Σ by $-1/K_x$. In (3.4), corrections of order $1/x^2$ do not arise either, because, recalling that ϕ_a contains the P and Q part, in (2.12), the P component is cancelled exactly, and any mixing of the Q part automatically increase the power of x by at least one in the denominator. The cancellation proven here is up to order $1/x^3$, and thus may affect the adiabatic potential. But certainly the PCD of the type $1/x$ has dropped out. This is the main result of this paper, showing that the PCD problem disappears if the full nonadiabatic corrections are taken into account. Evidently, this is valid if the rest of the contribution represented by G_{KK}^Σ is short ranged. This is shown next.

We stress the point that the cancellation (3.4) is possible only in the case where nearly the full spectrum is represented by G^Σ. Evidently, this does not happen in all the other cases where $\Sigma H \Sigma > 0$ and $E \simeq 0$ is not operative.

3.2 The higher-order corrections

Now consider the last correction term associated with G_{KK}^Σ. Explicitly,

$$\begin{aligned} U_a^{na-II} &= < K_x \Sigma G_K^\Sigma (H_{ad} - E_a) G^\Sigma \Sigma K_x > \\ &\equiv < K_x G_K^\Sigma \Sigma [H_{ad} - E_a] \Sigma \Omega^\Sigma >, \end{aligned} \tag{3.5}$$

where the new function $\Omega^\Sigma = \Sigma G^\Sigma K_x \phi_a$. The most important factor in (3.5) is the operator $(H_{ad} - E_a)$, which wipes out any state that is proportional to Π and thus P at large x, even without the projection Σ. Therefore, the projection Σ is fully operative there. In (3.5), we replaced G_K^Σ by the full G^Σ to make the expression symmetric. (We can equally replace G^Σ by G_K^Σ, and the argument goes through in a similar way.)

We note that the last operator ΣK_x on the right in (3.5) acts on all functions to its right. Therefore,

$$\begin{aligned} \Sigma K_x(\phi_a u_a(\vec{x})) = \Sigma[K_x\phi_a]u_a - (\hbar^2/M_{red})\Sigma[\nabla_x \phi_a] \\ \bullet \nabla_x u_a + \Sigma \phi_a K_x u_a(x). \end{aligned} \tag{3.6}$$

Obviously the last term is zero because Σ is orthogonal to ϕ_a. The first term can have contributions from the Q dependent part of the function, ϕ_a^Q, which results in $1/x^4$, because it is to be projected onto the orthogonal space Σ, and $\Sigma \to Q$ as $x \to \infty$. The second term is estimated to be of order $1/x^3$ or higher, because $\Sigma \nabla_x \phi_a \simeq \Sigma \nabla_x \phi_a^Q \to 1/x^3$; the ϕ_a^P contribution is zero. Note also that $\Omega^\Sigma \to \phi_a^Q \simeq G_a^{ad} V_1 \psi_a$ of section 2, where $\Omega_a \to Q\, 1/x^2$. Thus, we have, using the power-counting of sec. 2.3,

$$\begin{aligned} &< K_x(-1/K_x)[H_{ad} - E_a] G^\Sigma K_x > \simeq \\ &(1/x^2)(constant)(1/x^4)x^3 \simeq 1/x^3, \end{aligned} \tag{3.7}$$

where the first factor is from ϕ_a^Q, the second factor of constant is from $\Sigma(H_{ad} - E_a)\Sigma$, the third factor from $K_x \phi_a^Q$ using (2.12) and (2.16), and finally the x^3 factor from the volume integration. This implies that most of the Σ functions are exponentially decaying at large x, because the y component may oscillate as the x component exponentially decays. Therefore, U_a^{na-II} decays faster than any inverse power of x at large x.

This completes the proof of the Efimov effect, with

$$U_a(x) \to -const/x^2 + O(1/x^3) \tag{3.8}$$

In fact, this result is consistent with the fact that, when the full set of $\{\phi_a\}$ is taken into account in the derivation of the effective potential, it does not matter what kind of coordinate is chosen, and that the direct simple proof given in the Appendix A also supports this result. Incidentally, with the full potential given by (3.1), the restriction on the mass ratios has also disappeared.

An alternate proof of the result of this section in the $(\vec{y}, \vec{x})$ variables is given in Appendix B, where the separation between the two nonadiabatic corrections is not necessary. Therefore, the PCD never appears, and the leading correction is again found to be of order $1/x^3$.

4 Presence of two-body bound states

In previous works, it was always assumed that the two-body pair potential yields one bound state at the threshold. It is possible that the potential may be much stronger than this, so that, in addition to the one at zero energy, there may be one or more additional bound states with negative energies. In fact, in many practical cases, this

may be the more prevalent situations. We examine therefore the effect of a bound state below the threshold on the Effimov effect.

In the case when there is one bound state of the two-particle subsystem with $E_b < 0$, in addition to the zero energy bound state, then Φ^Σ of (3.6) is no longer a decaying function in x, although it decays in variable y. Besides, the Σ projection in the Green's function $1/[\Sigma(-K_x)\Sigma]^{-1}$ cannot be dropped. Thus, instead of (3.4), we have

$$G_K^\Sigma \simeq |\Phi_b)\ (E-H)^{-1}\ (\Phi_b| + G_{K'}^\Sigma \tag{4.1}$$

where the first term on the right hand side represents the contribution of the bound state to the Green's function, and thus is of short-range. The second term on the right of (4.1) is the same as that used in (3.2), so that we can carry out the analysis with the second part $G_{K'}^\Sigma$ in the same way.

5 Conclusion

We have completed the proof of the Efimov effect in the adiabatic representation by taking into account all the nonadiabatic corrections. This is of course as expected, since the earlier proofs given in [1] and [2] are valid and are not disputed here. However, because of the usefulness of the adiabatic treatment of the problem in many specific cases, as initiated in [3], it is important that the nonadiabatic corrections $< K_x >$ and $< K_x G K_x >$ are fully analyzed, so that the lowest adiabatic picture may be used with confidence.

It is therefore shown that, when all the nonadiabatic corrections are included, the adiabatic potential retains the correct leading behavior of $U_a^{ad} \simeq -1/x^2$ for the Efimov effect as the pair scattering length $A \to \infty$, and more importantly, the nonadiabatic correction behaves as $1/x^3$ or stronger. The effect of the presence of bound states below the threshold was also considered. An alternative view based on the static expansion is given in Appendix A. A more compact proof of the result of Sect. 3 in the adiabatic representation is given in Appendix B, where the details of the adiabatic solution are used. They further clarify the situation with respect to the adiabatic picture. In fact, the static picture adopted in Appendix A is simple, but seems to have the undesirable feature that the nonstatic correction also adds to the attractive dipole potential. By contrast, the adiabatic potential has the complete dipole contribution.

The PCD is conjectured to be caused by the unnatural freezing of particle 1 or particle 2 in the asymptotic region where, for example, the pair $1+3$ is bound, with near zero binding energy. A slightly different coordinate system may eliminate the PCD, while the proper particle translation factor that is multiplied to the overall wave function may also remove the difficulty. Such procedures are known to rectify the problems similar to the PCD in ion-atom collisions. These points will be elaborated on in future reports [5, 6]

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Appendix A. Efimove effect in the static representation

Analysis of the Efimov effect using the undistorted static state representation seems to be more direct and intuitive, and may further clarify the disastrous PCD problem associated with the nonadiabatic corrections in the adiabatic representation. Again, the proof is complete only if the higher order corrections are also properly estimated as to their long range contribution. In this appendix, $| \ > = |\psi_0 >$ of undistorted basis.

(i) The choice of coordinates. Unlike with the adiabatic case, it is not necessary to pick the $(\vec{x}, \vec{y})$ set. Instead, we take $\vec{r} = \vec{r}_{23}$ for the $2+3$ pair, and let $\vec{R}$ = the coordinate of particle 1 relative to the center of mass of the pair $(2+3)$. Then,

$$\begin{aligned} H &= K_R + (K_r + V_{23}) + (V_{12} + V_{13}) \\ &= K_R + H_r + V_1. \end{aligned} \tag{A.1}$$

An undistorted basis set is defined by

$$(h_r - E_n^{23})\psi_n^{23}(r) = 0 \tag{A.2}$$

where ψ^{23} are slightly different from $\psi^{(2)}(\vec{y})$ defined in Sect. 2 in the use of variable $\vec{r}$ and in the reduced mass. Compared with the adiabatic states generated by H_{ad} of (2.1), h_r is defined with mass μ_2 and contains no distortion via V_1. Obviously, this choice is not symmetric in particles 1 and 2, but does not affect the main result of this section. A symmetric treatment between all the particles may be given using the formalism of [7], if necessary. The treatment in Sects. 2 and 3 is for heavy particles 1 and 2, to which the result of this Appendix is to be compared.

(ii) The scattering equation for the ψ_0 component of the wave function can be derived in a similar fashion as in Sect. 2, and is given by

$$[K_R + \ < V_1 > \ + \ < V_1 G^Q V_1 > \ - E_{0'}]u_0(\vec{R}) = 0 \tag{A.3}$$

where Q = 1 − P, with P = $|\psi_0(\vec{r})\rangle\ (\psi_0(\vec{r}')|$ in terms of the undistorted functions, and

$$< V_1 >= \int \psi_0^+ V_1(\vec{r}, \vec{R})\psi_0(\vec{r})d\vec{r} \tag{A.4}$$

Since we are going to identify ψ_0 as the zero-energy 'bound state' function, with $E_0^{23} \simeq 0$, the corresponding wave function at large r may be written as

$$\psi_0 \to (r - A)/r, \text{ as } r \to \infty \tag{A.5}$$

where A is the scattering length. Eventually, for the zero energy bound state for the pair $(2+3)$, this parameter A approaches infinity.

Without loss of generality, we may assume that V_1 is of short-range, of δ function form. We have $\vec{r}_{12} = (M_3/(M_2+M_3))\vec{r} + \vec{R}$ and $\vec{r}_{13} = (M_2/(M_2+M_3))\vec{r} - \vec{R}$. Then, for $r \ll A$,

$$\begin{aligned} < V_1 > &= < \psi_0|V_1|\psi_0 > \\ &\simeq -\gamma A^2 \int_0^A d\vec{r}[\delta(\vec{r}_{12} + \delta(\vec{r}_{13})]/r^2 \\ &\simeq -\gamma' A^2/R^2, \end{aligned} \tag{A.6}$$

as $R \to \infty$ but for $R < A$, and where γ is the potential strength and γ' contains additional constant factors. The $-1/R^2$ behavior of (A.6) gives, when substituted into (A.1), the Efimov effect with an infinite number of bound states in the limit of large A.

(iii) To complete the proof, we have to consider the correction term in (A.3). In the present case, however, this is trivial, because, if there are no negative energy bound states of H other than the zero energy state associated with the Efimov potential, which are in the P space, then

$$\begin{aligned} < V_1 G^Q V_1 > &\equiv < \psi_0|V_1 G^Q V_1|\psi_0 > \\ &\simeq -(constant)/R^2 < 0 \end{aligned} \tag{A.7}$$

which follows from the fact that G^Q is negative definite and is bounded from above (because E − QHQ < 0). A rough closure approximation may then be used, in which case the short-range V_1 with the ψ_0 of the form (A.5) gives the above behavior. This in turn implies that the long-range potential of (A.6) is already strongly attractive enough to produce an infinite number of bound states. By Hylleraas-Undheim theorem, then, all the Efimov bound states will be made deeper by the additional attractive potential of (A.7).

Incidentally, it is interesting to note that in the static representation, the $-1/R^2$ first order potential is augmented by an additional contribution of (A.7), while in the adiabatic representation, the result of Sect. 3 and Appendix A shows that the entire $1/x^2$ comes from the adiabatic term. This suggests that for the Efimov studies, the adiabatic picture may be more favorable.

In case of one or more bound states (resonance states) present in the Q-space spanned by QHQ, we can subtract their contribution from G^Q. On the other hand, the subtracted terms are by definition of a short-range, and does not affect the long range behavior of interest here. Therefore, we have completed the proof.

Incidentally, the form (A.7) may be compared with the adiabatic potential, which is written in the undistorted state representation of (2.12a), as [8]

$$U_a^{ad} \simeq < V_1 >_a \; + \; < V_1 G_a^{ad} V_1 >_a \tag{A.8}$$

$$G_a^{ad} = [Q(\eta_a(R) - H_{ad}(\vec{r}, \vec{R}))Q]^{-1} \tag{A.9}$$

where we neglected the differences in the coordinates $(\vec{x}, \vec{y})$ and $(\vec{R}, \vec{r})$. The form (A.8) with (A.9) indicates that the main content of the potential U_a^{ad} is the first term in (A.8), which is just the static potential of (A.6).

Appendix B. An alternate proof of the nonadiabatic correction

In Sect. 3, we presented a direct proof of the cancellation of the PCD by a long range behavior of the nonadiabatic corrections. In this appendix, we give an alternate proof using the same coordinates $(\vec{y}, \vec{x})$, but first analyzing the contents of the adiabatic solution ϕ_a and the potential U_a^{ad} in terms of the 'static' basis set $\{\psi^{(2)}\}$ introduced in Sect. 2. In some sense, the approach here is similar to that of Appendix A.

The adiabatic potential and the wave function defined by (2.3) have the physical contents (2.12) and

$$U_a^{ad} = (\psi_a^{(2)}|V_1|\psi_a^{(2)}) + (\psi_a^{(2)}|V_1 G_a^{ad} V_1|\psi_a^{(2)}) \tag{B.1}$$

where $G_a^{ad} = [Q(\eta_a(x) - H_{ad})Q]^{-1}$. This is a nonlinear equation for U_a^{ad}, since G_a^{ad} contains this quantity through $\eta_a(x)$. It is obtained by projecting the adiabatic equation (2.3) onto the undistorted states, with $P^{(2)} = |\psi_a^{(2)})\,(\psi_a^{(2)}|$ and $Q^{(2)} = 1_{\vec{y}} - P^{(2)}$. The resulting coupled equations (algebraic, with $\vec{x}$ fixed) are then solved for the coefficients of ϕ_a^P and ϕ_a^Q of (2.12). This expansion shows the physical contents of the adiabatic solution. The expression (B.1) is exact in the adiabatic approximation, for the coordinates $(\vec{y}, \vec{x})$.

We now compare (B.1) with the exact effective potential U_a for the state 'a'. The exact effective potential can be written, at least formally, as

$$U_a = (\psi_a^{(2)}|V_1|\psi_a^{(2)}) + (\psi_a^{(2)}|V_1 G^Q V_1|\psi_a^{(2)}) \tag{B.2}$$

where $G^Q = [Q(E-H)Q]^{-1}$ and $H = K_x + H_{ad}$. Evidently, G^Q is a very difficult many-particle operator to evaluate, but can be used below for the limited purpose of estimating the asymptotic behavior at large x. The nonadiabatic correction is given by the difference between the two second terms in (B.1) and (B.2),

$$\begin{aligned} U_a - U_a^{ad} \equiv U_a^{na} &= (\psi_a^{(2)}|V_1[G^Q - G_a^{ad}]V_1|\psi_a^{(2)}) \\ &= -(\psi_a^{(2)}|V_1 G_a^{ad}[(E_a - \eta_a(x)) - K_x] \\ &\quad \times G_a^Q V_1|\psi_a^{(2)}) \end{aligned} \tag{B.3}$$

Noting that $G_a^{ad} V_1 \psi_a^{(1)} \simeq \phi_a^Q$ from (2.12) and making the replacement $G_a^Q \simeq G_a^{ad}$, we have

$$U_a^{na} \to -(\phi_a^Q|\phi_a^Q)/x^2 + (\phi_a^Q|K_x|\phi_a^Q), \tag{B.4}$$

where we used $E_a - \eta_a(x) \simeq 1/x^2$ in the first term. From Sect. 2, $\phi_a^Q \simeq 1/x^2$, so that

$$(\phi_a^Q|\phi_a^Q)/x^2 \to (1/x^2)^2 x^3/x^2 \simeq 1/x^3 \tag{B.5}$$

Furthermore, the second term behaves at large x as

$$(\phi_a^Q|K_x|\phi_a^Q) \to (1/x^2)(1/x^4)x^3 \simeq 1/x^3 \tag{B.6}$$

Therefore, we have finally

$$U_a^{na} \simeq 1/x^3. \tag{B.7}$$

This completes the analysis of the potential for the two heavy particles in the adiabatic representation, using the 'static' basis and with the coordinates $(\vec{y}, \vec{x})$. In fact, the proof given above seems to be independent of the coordinates chosen; i.e. either $(\vec{y}, \vec{x})$ which was used here, or $(\vec{r}, \vec{R})$ used in Appendix A may be acceptable. This point is relevant in the work of [5], where the latter coordinates are used to demonstrate the absence of PCD. The result we obtained here is consistent with that in Sect. 3.

The advantage of the approach adopted in this Appendix B is that the nonadiabatic correction is obtained in one step, without separating it into two parts as was done in Sect. 3. Evidently, the separation into parts I and II gave rise to the PCD difficulty, although it is the natural progression when the adiabatic set $\{\phi_a\}$ is employed, with the heavy particle kinetic energy operator K_x as the perturbation that mixes states of different α's.

Finally, we make a remark on the zero-energy bound state (ZEBS) of interest here for the Efimov effect, and on the projection operators Π and Σ. The ZEBS is a very special transitional state [9], with tricky asymptotic behavior. Depending on the limits $k_\pm \to 0$, where the $k_\pm$ are the complex momenta of the pair, one has either a normalizable wave function or a non-normalizable continuum function. In fact, the ZEBS is 'half-way' between the bound and scattering states, as manifested by the Levinson's theorem

$$\delta(E = 0) = (m + 1/2)\pi \tag{B.8}$$

where δ is the phase shift at E = 0 and m is a finite integer for the number of bound states with negative binding energy. The remaining contribution of $\pi/2$ corresponds to that for the ZEBS. Therefore, the distinction between the Π and Σ becomes meaningless when E = 0. This fact is used in (3.4) in the cancellation and disappearence of the PCD. Of course, the analysis in this Appendix shows that such an anomaly and cancellation is an artifact of the theory in which the nonadiabatic corrections are split up into two parts, although in terms of K_x this is a natural thing to do.

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